MOMENTS OF THE GENERAL EPIDEMIC MODEL

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1. Introduction

In this note we outline an elementary approach to the problem of the general stochastic epidemic model. Our method makes it possible to calculate as many moments as may be required of the stochastic variables involved. It takes into account finite population size, N, and indeed the moments are obtained as expansions in powers of $\frac{1}{N}$.

The corresponding problem even for the simple stochastic epidemic model is non-trivial since the standard exact solution does not lend itself to effective calculation unless N is very small. Some authors [(1), (2), (3)] have instead developed asymptotic expansions for the distribution in descending powers of N, the simplest method being that of Weiss (3). Although the functions obtained are rather complicated and only of limited direct use, they do make it possible to derive the first few terms of the asymptotic expansions of as many moments as we may reasonably be interested in.

It does not seem possible, however, to extend any of these methods to

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the general epidemic model. We therefore propose a very simple new approach which by-passes the distribution functions and calculates the moments as power series in $\frac{1}{N}$, directly from the basic recurrence relations. To make our method clear we shall first demonstrate its application to the simple epidemic.

It will be noted that the expansions obtained for both models can be useful only so long as the expected number of infected individuals is much smaller than the total population. In fact the same limitation holds also for the results of the previous authors we have quoted.

2. The Simple Epidemic

Assume a population of $N+1$ individuals of whom, initially, one is infected and the remainder susceptible. At any subsequent time, t, the numbers of infected and susceptible individuals are denoted by $x(t)$, $y(t)$ respectively. It is assumed that infected cases remain so indefinitely, and that none are isolated or otherwise removed from the population. The probability of an additional infection occurring in the time interval $(t,t+\delta t)$ is assumed to be $\beta x(t)y(t)\delta t+o(\delta t)$ where β is a constant. If

$$p_n(t) = \Pr[x(t)=n] , \tag{1}$$

then we define the generating function

$$G(\lambda,t) = \sum_{n=0}^{\infty} \lambda^n p_n(t) . \tag{2}$$

Following a standard method [cf(4), p. 73] and remembering that $y=N+1-x$, we see that $G(\lambda,t)$ satisfies the equation

$$\frac{\partial G}{\partial t} = (\lambda-1)\cdot\beta\lambda \frac{\partial}{\partial\lambda} (N+1-\lambda \frac{\partial}{\partial\lambda})G = \beta N\lambda(\lambda-1)(\frac{\partial G}{\partial\lambda} - \frac{\lambda}{N} \frac{\partial^2 G}{\partial\lambda^2}) . \tag{3}$$

We set $\tau = Nt$ and obtain

$$\frac{\partial G}{\partial\tau} = \lambda(\lambda-1)\{\frac{\partial G}{\partial\lambda} - \frac{\lambda}{N} \frac{\partial^2 G}{\partial\lambda^2}\} . \tag{4}$$

Consider the factorial moments of $x(t)$. For this purpose we define

$$(x)_r = x(x-1)(x-2)\cdots(x-r+1) \qquad (r=1,2,\cdots) \tag{5}$$

$$(x)_0 = 1 \tag{6}$$

and

$$M_r(\tau) = E[(x)_r] \; . \tag{7}$$

If we write $G_r = \frac{\partial^r G}{\partial \lambda^r}$, then

$$M_r(\tau) = G_r(1,\tau) \; . \tag{8}$$

Now differentiate (4) r times, using Leibniz' theorem, and set $\lambda=1$. This yields

$$\frac{dM_r}{d\tau} = rM_r + r(r-1)M_{r-1} - \frac{1}{N}\{rM_{r+1} + 2r(r-1)M_r + r(r-1)(r-2)M_{r-1}\} \; , \tag{9}$$

with

$$M_0(0) = M_1(0) = 1 \quad , \quad M_r(0) = 0 \qquad (r\geqslant 2) \; . \tag{10}$$

The next step is to set

$$M_r(\tau) = \mu_r(\tau) + \frac{A_r(\tau)}{N} + \frac{B_r(\tau)}{N^2} + \cdots \tag{11}$$

and substitute in (9). Then

$$\frac{d}{d\tau}\{\mu_r + \frac{A_r}{N} + \frac{B_r}{N^2} + \cdots\} = r\{\mu_r + \frac{A_r}{N} + \frac{B_r}{N^2} + \cdots\} + r(r-1)\{\mu_{r-1} + \frac{A_{r-1}}{N} + \cdots\}$$

$$- \frac{1}{N}\{r(\mu_{r+1} + \frac{A_{r+1}}{N} + \cdots) + 2r(r-1)(\mu_r + \frac{A_r}{N} + \cdots)$$

$$+ r(r-1)(r-2)(\mu_{r-1} + \frac{A_{r-1}}{N} + \frac{B_{r-1}}{N^2} + \cdots)\} \ . \qquad (12)$$

Comparing like powers of $\frac{1}{N}$, we get

$$\frac{d\mu_r}{d\tau} = r\mu_r + r(r-1)\mu_{r-1} \qquad (13)$$

$$\frac{dA_r}{d\tau} = rA_r + r(r-1)A_{r-1} - [r\mu_{r+1} + 2r(r-1)\mu_r + r(r-1)(r-2)\mu_{r-1}] \qquad (14)$$

$$\frac{dB_r}{d\tau} = rB_r + r(r-1)B_{r-1} - \{rA_{r+1} + 2r(r-1)A_r + r(r-1)(r-2)A_{r-1}\} \qquad (15)$$

$$\vdots$$

with initial conditions

$$\mu_0(0) = \mu_1(0) = 1$$

$$\mu_r(0) = 0 \qquad (r \geq 2) \qquad (16)$$

$$A_r(0) = B_r(0) = \cdots = 0 \qquad (r=0,1,2,\cdots)$$

Equations (13) with initial conditions (16) can be solved directly and lead to

$$\mu_r(\tau) = r!e^{\tau}(e^{\tau}-1)^{r-1} \qquad (r \geq 1) . \tag{17}$$

Substituting in (14) for $r=1,2$ we get

$$\frac{dA_1}{d\tau} = A_1 - 2e^{\tau}(e^{\tau}-1) \tag{18}$$

$$\frac{dA_2}{d\tau} = A_2 + 2A_1 - 4e^{\tau}(e^{\tau}-1)(3e^{\tau}-1) \tag{19}$$

with $A_1(0)=A_2(0)=0$, and hence

$$\begin{aligned} A_1(\tau) &= -2e^{2\tau} + 2e^{\tau}(1+\tau) \\ A_2(\tau) &= 4e^{\tau}(\tau+1) + 4e^{2\tau}(3\tau+2) - 12e^{3\tau} . \end{aligned} \tag{20}$$

The procedure can clearly be continued to higher values of r and the coefficients $B_r(\tau)$ can also be calculated.

If $m_r(\tau)$ denotes the expectation of $[x(\tau)]^r$, then

$$m_1(\tau) = M_1(\tau) = e^{\tau} + \frac{1}{N}[2e^{\tau}(1+\tau)-2e^{2\tau}] + O(\frac{1}{N^2}) \tag{21}$$

$$m_2(\tau) = M_2(\tau) + M_1(\tau) = 2e^{2\tau} - e^{\tau} + \frac{1}{N}[6e^{\tau}(\tau+1)+6e^{2\tau}(2\tau+1)+12e^{3\tau}] + O(\frac{1}{N^2}) . \tag{22}$$

We see, as mentioned above, that these expansions are useful only so long as $e^{\tau} \ll N$.

3. The General Epidemic Model

In this model it is assumed that an infected individual, when diagnosed, is isolated from the population and subsequently, whether following recovery, death or for any other reason, is immune to the infection. Again start with a population of $N+1$ individuals of whom one is infected and N are susceptible. We shall later deal with the case where the initial number of infected individuals is different from 1. At any time t, let

$x(t)$ = number of susceptible individuals;

$y(t)$ = number of infected individuals;

$z(t)$ = number of those who have been isolated following infection.

It is assumed that, in the interval $(t,t+\delta t)$ there is probability $\beta xy\delta t+o(\delta t)$ of a new infection (i.e. that the set x,y,z changes to $x-1,y+1,z$) and $\gamma y\delta t+o(\delta t)$ that an infected individual is diagnosed and isolated (i.e., that x,y,z changes to $x,y-1,z+1$). The probability of other types of change, e.g., multiple infections, is assumed to be $o(\delta t)$. Clearly

$$x + y + z = N + 1 \qquad (t \geq 0) \tag{23}$$

while $x(0)=N$, $y(0)=1$, $z(0)=0$.

We take y, z as our stochastic variables, and define

$$P_{r,s}(t) = P_r[y(t)=r \ , \ z(t)=s]$$

and

$$G(\eta,\zeta,t) = \sum_{r,s=0}^{\infty} \eta^r \zeta^s P_{r,s}(t) \ . \tag{24}$$

Since the possible transitions in the time-interval $(t,t+\delta t)$ are:

$$(y,z) \to (y+1,z) \quad \text{with probability} \quad \beta y(N+1-y-z)\delta t + o(\delta t)$$

and

$$(y,z) \to (y-1,z+1) \quad \text{with probability} \quad \gamma y \delta t + o(\delta t)$$

it follows that [cf (4), p. 73]

$$\begin{aligned}\frac{\partial G}{\partial t} &= (\eta-1)\beta\eta \frac{\partial}{\partial\eta} (N+1-\eta \frac{\partial}{\partial\eta} -\zeta \frac{\partial}{\partial\zeta})G + (\frac{\zeta}{\eta} -1)\cdot\gamma\eta \frac{\partial G}{\partial\eta} \\ &= \beta\eta(\eta-1)(N \frac{\partial G}{\partial\eta} -\eta \frac{\partial^2 G}{\partial\eta^2} -\zeta \frac{\partial^2 G}{\partial\eta\partial\zeta}) + \gamma(\zeta-\eta) \frac{\partial G}{\partial\eta} \ .\end{aligned} \tag{25}$$

Now let

$$G_{h,k}(\eta,\zeta,t) = \frac{\partial^{h+k} G}{\partial\eta^h \partial\zeta^k} \tag{26}$$

and

$$M_{h,k(t)} = G_{h,k}(1,1,t) \ . \tag{27}$$

Clearly $M_{h,k}(t)$ is the joint factional moment of order h,k , i.e. $E[(y)_h(z)_k]$.

If $\tau=\beta Nt$, $\rho=\gamma/\beta$ then (25) becomes

$$\frac{\partial G_{0,0}}{\partial\tau} = \eta(\eta-1)G_{1,0} - \frac{1}{N}\{\eta^2(\eta-1)G_{2,0} + \eta\zeta(\eta-1)G_{1,1} + \rho(\eta-\zeta)G_{1,0} \tag{28}$$

and hence, differentiating and setting $\eta=\zeta=1$,

$$\begin{aligned}\frac{\partial M_{h,k}}{\partial\tau} = hM_{h,k} + h(h-1)M_{k-1,k} - \frac{1}{N}\{&hM_{h+1,k} + hM_{h,k+1} + h[2(h-1)+k+\rho]M_{h,k} \\ &+ h(h-1)M_{h-1,k+1} - \rho kM_{h+1,k-1} + h(h-1)(h+k-2)M_{h-1,k}\} \ .\end{aligned} \tag{29}$$

We now write

$$M_{h,k}(\tau) = \mu_{h,k} + \frac{A_{h,k}}{N} + \frac{B_{h,k}}{N^2} + \cdots \ . \tag{30}$$

If we substitute this in (29), and compare like powers of $\frac{1}{N}$, we get, after a little manipulation,

$$\frac{d\mu_{h,k}}{dt} = h\mu_{h,k} + h(h-1)\mu_{h-1,k} \tag{31}$$

$$\begin{aligned}\frac{dA_{h,k}}{d\tau} = hA_{h,k} + h(h-1)A_{h-1,k} &- h\mu_{h+1,k} - h\mu_{h,k+1} - h[2(h-1)+k+\rho]\mu_{n,k} \\ &- h(h-1)\mu_{h-1,k+1} + \rho k\mu_{h+1,k-1} - h(h-1)(h+k-2)\mu_{h-1,k}\end{aligned} \tag{32}$$

$$\frac{dB_{h,k}}{d\tau} = hB_{h,k} + h(h-1)B_{h-1,k} - hA_{h+1,k} - hA_{h,k+1} - h[2(h-1)+k+\rho]A_{h,k}$$

$$- h(h-1)A_{h-1,k+1} + \rho kA_{h+1,k-2} - h(h-1)(h+k+2)A_{h-1,k} \; . \qquad (33)$$

The initial conditions are

$$\mu_{0,0}(0) = \mu_{1,0}(0) = 1$$

$$\mu_{h,k}(0) = 0 \qquad \text{(all other values of } h,k) \qquad (34)$$

$$A_{h,k}(0) = B_{h,k}(0) = 0 \qquad (h,k=0,1,2,\cdots)$$

The system (31) with initial conditions can be solved as before and leads to

$$\mu_{h,k}(\tau) = \delta_{k,0}h!e^{\tau}(e^{\tau}-1)^{h-1} \qquad (35)$$

showing incidentally that the first moment of $\zeta(\tau)$ is $O(\frac{1}{N})$.

Now consider some cases of (32).

(i) $h=1$, $k=0$: $\frac{dA_{1,0}}{d\tau} = A_{1,0} - \mu_{2,0} - \mu_{1,1} - \rho\mu_{1,0}$ (36)

with $A_{1,0}(0)=0$,

yielding $A_{1,0}(\tau) = [(2-\rho)\tau+2]e^{\tau} - 2e^{2\tau}$. (37)

(ii) $h=0$, $k>0$: $\frac{dA_{0,k}}{d\tau} = \rho k\mu_{1,k-1} = \rho k\delta_{k,1}e^{\tau}$ (38)

with $A_{0,k}(0)=0$

from which, by (35),

$$A_{0,k}(\tau) = \rho k \delta_{k,1}(e^{\tau}-1) . \tag{39}$$

(iii) $h=2$, $k=0$: A similar elementary argument leads to

$$A_{2,0}(\tau) = -2[(2-\rho)\tau+2+\rho]e^{\tau} + 2[2(3-\rho)\tau+8+\rho]e^{2\tau} - 12e^{3\tau} . \tag{40}$$

(iv) $h=k=1$: In this case we get

$$A_{1,1}(\tau) = 2\rho e^{2\tau} - 2\rho(\tau+1)e^{\tau} . \tag{41}$$

The system (33) can be treated similarly and we get

$$B_{0,1}(\tau) = \rho(\tau-1)(4-\rho)e^{\tau} - e^{2\tau} + 5\rho - \rho^2$$

$$B_{0,2}(\tau) = 2\rho^2 e^{2\tau} - 4\rho^2\tau e^{\tau} - 2\rho^2$$

and $B_{1,1}(\tau)$, $B_{2,0}(\tau)$, $B_{1,0}(\tau)$ can be calculated similarly.

4. Results

Combining the results obtained as above we finally obtain

$$\langle y\rangle = M_{1,0} = e^{\tau} - \frac{1}{N}\{2e^{2\tau}+[(\rho-2)\tau-2]e^{\tau}\} + O(\frac{1}{N^2})$$

$$\langle y^2\rangle = M_{2,0} + M_{1,0}$$

$$= 2e^{2\tau} - e^{\tau} - \frac{e^{\tau}}{N}\{[(2-\rho)\tau+2(\rho+1)]-2e^{\tau}[2(3-\rho)\tau-\rho-7]+12e^{2\tau}\} + O(\frac{1}{N^2})$$

$$\langle yz\rangle = M_{1,1} = \frac{2\rho e^{\tau}}{N}\{e^{\tau}-(\tau+1)\} + O(\frac{1}{N^2})$$

$$\langle z\rangle = M_{0,1} = \frac{\rho}{N}(e^{\tau}-1) + \frac{\rho}{N^2}\{(\tau-1)(4-\rho)e^{\tau}-e^{2\tau}+5-\rho\} + O(\frac{1}{N^3})$$

$$\langle z^2\rangle = M_{0,1} + M_{0,2}$$

$$= \frac{\rho}{N}(e^{\tau}-1) + \frac{\rho}{N^2}\{[(\tau-1)(4-\rho)-4\rho\tau]e^{\tau}+(2\rho-1)e^{2\tau}+5-3\rho\} + O(\frac{1}{N^3}) .$$

Various other statistics, including correlation coefficients can all be deduced from these.

5. More General Initial Conditions

If we assume as initial conditions

$$x(0) = N$$

$$y(0) = a$$

$$z(0) = 0$$

where a is any natural number then the differential equations are unchanged but the initial conditions become

$$\mu_{r,s}(0) = \delta_{s,0}(a)_r \qquad (r=1,2,\cdots)$$

$$A_{r,s}(0) = 0 \qquad (r,s=0,1,\cdots) .$$

Since no new principle is involved we shall merely quote here one typical result, viz.

$$M_{1,0}(\tau) = \langle y \rangle = ae^{\tau} - \frac{ae^{\tau}}{N}\{[(\rho-2)\tau-(a+1)]+(a+1)e^{\tau}\} + 0(\frac{1}{N^2}) .$$

6. Some Numerical Results

We see from (17) that in the absence of any procedure for removing infected individuals and on the assumption of an infinitely large population, the expected number of infected individuals is e^{τ}. To illustrate the effect of removals and of finite N we write

$$M_{1,0}(\tau) = e^{\tau} - \Delta$$

and list some values below.

	ρ	Δ
	0	18.9
$N = 1000$	1	19.4
$e^{\tau} = 100$	10	23.5
	50	41.9

	ρ	Δ (e^{τ}=500)	Δ (e^{τ}=1000)
	0	49.3	198
$N = 10{,}000$	10	52.4	205
	100	80.3	267

References

(1) D.R. McNeil, Biometrika 59, 494 (1972).

(2) N.T.J. Bailey, Biometrika 55, 199 (1968).

(3) G.H. Weiss, (private communication).

(4) N.T.J. Bailey, The Elements of Stochastic Processes (Wiley, New York, 1964).